JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **37**, No. 4, November 2024
http://dx.doi.org/10.14403/jcms.2024.37.4.223

# ON EXTENDING $DC_k^p$-STRUCTURES

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Abstract. It is known that the concept of $DC_k$-spaces for any integer $k \geq 1$ and $k = \infty$ which is located at an intermediate stage between the concepts of co-$H$-spaces and co-$T$-spaces, where $DC_\infty$-space is a co-$H$-space and $DC_1$-space is a co-$T$-space. Clearly, any co-$H$-space is a co-$T$-space. We show that a result about extending of $DC_k^p$-structure. As a corollary, we can obtain a result about extending of co-$T$-structure which is a generalization of Hilton, Mislin and Roitberg's result about extending of co-$H$-structure.

## 1. Introduction

A space $(X, \mu)$ is called a *co-H-space* if there is a *co-H-structure* $\mu : X \to X \vee X$ such that $j\mu \sim \Delta : X \to X \times X$, where $j : X \vee X \to X \times X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. Let $(X, \mu)$ and $(X', \mu')$ be co-$H$-spaces. Then a map $f : (X, \mu) \to (X', \mu')$ is called a *co-H-map* if $\mu' f \sim (f \vee f)\mu : X \to X' \vee X'$. In [1], Aguade introduced a $T$-space as a space $X$ having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. As a dual space of $T$-space, we introduced [14] that a space $X$ is called a *co-T-space* if there is a *co-T-structure* $\theta : X \to X \vee \Omega\Sigma X$ such that $j\theta \sim (1 \times e')\Delta : X \to X \times \Omega\Sigma X$, where $j : X \vee \Omega\Sigma X \to X \times \Omega\Sigma X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. It is easy to show that any co-$H$-space $(X, \mu)$ is a co-$T$-space $(X, \theta)$ taking $\theta = (1 \vee e')\mu$. Let $(X, \theta)$ and $(X', \theta')$ be co-$T$-spaces. Then a map $f : (X, \theta) \to (X', \theta')$ is called a *co-T-map* if $\theta' f \sim (f \vee \Omega\Sigma(f))\theta : X \to X' \vee \Omega\Sigma X'$. Clearly, any co-$H$-map $f : (X, \mu) \to (X', \mu')$ induce a co-$T$-map $f : (X, \theta) \to (X', \theta')$ from the fact that $(f \vee \Omega\Sigma(f))(1 \vee e'_X) \sim (1 \vee e'_{X'})(f \vee f)$, where $\theta = (1 \vee e'_X)\mu$ and $\theta' = (1 \vee e'_{X'})\mu'$. It is known [19] that the concept of $DC_k$-spaces for

Received November 01, 2024; Accepted November 30, 2024.
2020 Mathematics Subject Classification: 55P45, 55P35.
Key words and phrases: $DC_k^p$-structures, co-$T$-spaces.
This work was supported by Hannam University Research Fund during research year 2024.

any integer $k \geq 1$ and $k = \infty$ which is located at an intermediate stage between the concepts of co-$H$-spaces and co-$T$-spaces, where $DC_\infty$-space is a co-$H$-space and $DC_1$-space is a co-$T$-space. The concept of $DC_k$-spaces is generalized to the concepts of $DC_k^p$-spaces for a map $p : X \to A$ [20]. It is also known [20] that if $X$ is a $DC_m^p$−space, then $X$ is a $DC_n^p$−space for any $n \leq m$. In 1978, Hilton, Mislin and Roitberg showed [6] that if $X$ and $X'$ are co-$H$-spaces, and $r : X' \to X$ is a co-$H$-map, then there is a co-$H$-structure on $C_r$ such that $i_r : X \to C_r$ is a co-$H$-map. In this paper, we show that if $X$ is a $DC_k^p$-space with a $DC_k^p$-structure $\theta : X \to A \vee F_k^X$ and $X'$ is a $DC_k^{p'}$-space with a $DC_k^{p'}$-structure $\theta' : X' \to A' \vee F_k^{X'}$ and a map $(s, r) : p' \to p$ is a $DC_k$-map from $\theta'$ to $\theta$, then there is a $DC_k^{\bar{p}}$-structure $\bar{\theta} : C_r \to C_s \vee F_k^{C_r}$ for $C_r$ such that a map $(i_s, i_r) : \theta \to \bar{\theta}$ is a $DC_k$-map from $\theta$ to $\bar{\theta}$. Taking $p = 1_X$ and $k = 1$, we can obtain a result that if $X$ is a co-$T$-space with a co-$T$-structure $\theta : X \to X \vee \Omega\Sigma X$ and $X'$ a co-$T$-space with a co-$T$-structure $\theta' : X' \to X' \vee \Omega\Sigma X'$ and $r : X \to X'$ is a co-$T$-map, then there is a co-$T$-structure on $C_r$ such that $i_r : X \to C_r$ is a co-$T$-map. Thus the above result is a generalization of the above Hilton, Mislin and Roitberg's result about extending of co-$H$-structure.

Throughout this paper, space means a space of the homotopy type of 1-connected locally finite $CW$ complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$ and the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. $\Sigma X$ denote the reduced suspension of X and $\Omega X$ denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by $\tau$. The symbols $e$ and $e'$ denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively.

## 2. $DC_k^p$-spaces

Let $p : X \to A$ be a map. A based map $g : X \to B$ is called *p-cocyclic* [12] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ \Delta \big\downarrow & & j \big\downarrow \\ X \times X & \xrightarrow{(p \times g)} & A \times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map $\theta$ an *co-associated map* of an $p$-cocyclic map $g$. Clearly, $g$ is $p$-cocyclic if and only if $p$ is $g$-cocyclic. A map $p : X \to A$ is *cocyclic* [13] if there is a map $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. Clearly, a based map $g : X \to B$ is cocyclic if and only if $g$ is $1_X$-cocyclic. It is also clear that a space $X$ is a co-$H$-space if and only if the identity map $1_X$ of $X$ is cocyclic. We called a space $X$ as a *co-$H^p$-space for a map $p : X \to A$* [17] if there is a cocyclic map $p : X \to A$, that is, there is a co-$H^p$-structure $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. In [1], Aguade introduced a $T$-space as a space $X$ having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is shown [1] that $X$ is a $T$-space if and only if $e : \Sigma\Omega X \to X$ is cyclic. We [14] introduced a co-$T$-space, which is the dual of a $T$-space,as if $e' : X \to \Omega\Sigma X$ is cocyclic. Clearly, any co-$H$-space is a co-$T$-space. We [19] introduced the concept of $DC_k$-spaces for any integer $k \geq 1$ and $k = \infty$ which is located at an intermediate stage between co-$H$-spaces and co-$T$-spaces, where $DC_\infty$-space is a co-$H$-space and $DC_1$-space is a co-$T$-space. We [20] also generalized the concept of $DC_k$-spaces to the concepts of $DC_k^p$-spaces for a map $p : X \to A$. We called a space $X$ as a *co-$T^p$-space for a map $p : X \to A$* [18] if $e' : X \to \Omega\Sigma X$ is $p$-cocyclic, that is, there is a co-$T^p$-structure $\theta : X \to \Omega\Sigma X \vee A$ such that $j\theta \sim (e' \times p)\Delta$, where $j : \Omega\Sigma X \vee A \to \Omega\Sigma X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Clearly, a co-$T^1$-space is a co-$T$-space. *The dual Gottlieb set* denoted by $DG(X, B)$ is the set of all homotopy classes of cocyclic maps from $X$ to $B$. *The dual Gottlieb set for a map $p : X \to A$* [16], $DG^p(X, B) = DG(X, p, A; B)$, is the set of all homotopy classes of $p$-cocyclic maps from $X$ to $B$. In fact, a 1-cocyclic map is a cocyclic map. In general, $DG(X, B) \subset DG^p(X, B) \subset [X, B]$ for any map $p : X \to B$ and any space $B$. However, it is known [16] that $DG(S^n \times S^n, K(\mathbb{Z}, n)) \neq DG^{p_1}(S^n \times S^n, K(\mathbb{Z}, n)) \neq H^n(S^n \times S^n; \mathbb{Z})$. co-$T^p$-spaces are completely characterized by generalized dual Gottlieb sets. It is known [18] that for a map $p : X \to A$, $X$ is a co-$T^p$-space if and only if $DG^p(X, \Omega B) = [X, \Omega B]$ for any space $B$. Thus we obtain a result [14] that $X$ is a co-$T$-space if and only if $DG(X, \Omega B) = [X, \Omega B]$ for any space $B$. Ganea [4] introduced the concept of cocategory of a space as follows; Let $X$ be any space. Define a sequence of cofibrations

$$\mathcal{C}_k : X \xrightarrow{e'_k} F_k \xrightarrow{s'_k} B_k \ (k \geq 0)$$

as follows, let $\mathcal{C}_0 : X \overset{e'_0}{\to} cX \overset{s'_0}{\to} \Sigma X$ be the standard cofibration. Assuming $\mathcal{C}_k$ to be defined, let $F'_{k+1}$ be the fibre of $s'_k$ and $e''_{k+1} : X \to F'_{k+1}$ lift $e'_k$. Define $F_{k+1}$ as the reduced mapping cylinder of $e''_{k+1}$, let $e'_{k+1}$ : $X \to F_{k+1}$ is the obvious inclusion map, and let $B_{k+1} = F_{k+1}/e'_{k+1}(X)$ and $s'_{k+1} : F_{k+1} \to F_{k+1}/e'_{k+1}(X)$ the quotient map. The cocategory of $X$, [4] *cocat* $X$, is the least integer $k \geq 0$ for which there is a map $r : F_k \to X$ such that $r \circ e'_k \sim 1$. That is, *cocat* $X \leq k$ if and only if $e'_k : X \to F_k$ has a left homotopy inverse. It is well known [9] that *cocat* $X \leq 1$ if and only if $X$ is an $H$-space. Thus we know that *cocat* $\Omega B \leq 1$ for any space $B$. It is known [5]([7]) that $cat(X) \leq k$ if and only if $e_k : P^k(\Omega X) \to X$ has a right homotopy inverse. We can easily show that $F_1$ and $\Omega\Sigma X$ have the same homotopy type. A space $X$ is called a *$DC_k$-space* [19] if the inclusion $e'_k : X \to F_k$ is cocyclic, that is, there is a it $DC_k$-structure $\theta : X \to X \vee F_k$ such that $j\theta \sim (1 \times e'_k)\Delta$, where $j : X \vee F_k \to X \times F_k$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. It is shown [3] that *cocat* $Z \leq 1$ if and only if $Z$ can be dominated by a loop space. The property of the co-$T$-spaces is extended to the $DC_k$-spaces using cocategory. It is known [19] that $X$ is a $DC_k$-space if and only if $DG(X, B) = [X, B]$ for any space $B$ with *cocat* $B \leq k$. Thus we know that $X$ is a $DC_1$-space if and only if $X$ is a co-$T$-space. Let $p : X \to A$ be a map. A space $X$ is called a *$DC^p_k$-space* [20] if $e'_k : X \to F_k$ is $p$-cocyclic, that is, there is a map $\theta : X \to A \vee F_k$ such that $j\theta \sim (p \times e'_k)\Delta$, where $j : A \vee F_k \to A \times F_k$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. If $X$ is a $DC^p_k$-space, then we call such a map $\theta : X \to A \vee F_k$ a *$DC^p_k$-structure* for a space $X$. Clearly, a $DC^1_k$-space for the identity map $1 : X \to X$ is a $DC_k$-space [19]. $DC^p_k$-spaces are closely related by the dual Gottlieb sets for maps and cocategory of spaces. It is known [20] that for a map $p : X \to A$ be a map, a space $X$ is a $DC^p_k$-space if and only if $DG^p(X, Z) = [X, Z]$ for any space $Z$ with *cocat* $Z \leq k$. Moreover, it is also known [12] that if $f : X \to Y$ is $p$-cocyclic and $g : Y \to Z$ is any map, then $gf : X \to Z$ is $p$-cocyclic. Any co-$H$-space $X$ is a co-$H^p$-space and any co-$H^p$-space

$X$ is a co-$T^p$-space for any map $p : X \to A$. Thus we have the following results.

PROPOSITION 2.1. [20] *Let $p : X \to A$ be any map.*
*(1) $X$ is a $DC_1^p-$space $\Leftrightarrow$ $X$ is a co-$T^p$-space.*
*(2) If $X$ is a co-$H^p$-space, then for any $m \in N$, $X$ is a $DC_m^p$-space.*
*(3) If $X$ is a $DC_m^p-$space, then $X$ is a $DC_n^p-$space for any $n \leq m$.*
*(4) If $X$ is a $DC_k^p-$space and cocat $X \leq k$, then $X$ is a co-$H^p$-space.*

## 3. Extending $DC_k^p$-structures

Given maps $p : X \to A$, $p' : X' \to A'$, let $(s, r) : p' \to p$ be a map from $p'$ to $p$, that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{p'} & A' \\ r\downarrow & & s\downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \to C_r$ be the cofibration induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$. Let $i_s : A \to C_s$ be the cofibration induced by $s : A' \to A$ from $\iota_{A'} : A' \to cA'$. Then there is a map $\bar{p} : C_t \to C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{p} & A \\ i_r\downarrow & & i_s\downarrow \\ C_r & \xrightarrow{\bar{p}} & C_s, \end{array}$$

where $C_r = cX' \amalg X/[x', 1] \sim r(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{p} : C_r \to C_s$ is given by $\bar{p}([x', t]) = [p'(x'), t]$ if $[x', t] \in cX'$ and $\bar{p}(x) = p(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

DEFINITION 3.1. *Given maps $p : X \to A$, $p' : X' \to A'$, let $(s, r) : p' \to p$ be a map from $p'$ to $p$. Assume that $X'$ is a $DC_k^{p'}$-space with a $DC_k^{p'}$-structure $\theta' : X' \to A' \vee F_k^{X'}$ and $X$ is a $DC_k^p$-space with a $DC_k^p$-structure $\theta : X \to A \vee F_k^X$. Then we say that $(s, r) : p' \to p$ is a $DC_k$-map* *from $\theta'$ to $\theta$ if the following diagram is homotopy commutative;*

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & A' \vee F_k^{X'} \\ r\big\downarrow & & \big\downarrow (s\vee F_k(r)) \\ X & \xrightarrow{\theta} & A \vee F_k^X. \end{array}$$

The following lemmas are well known redults.

LEMMA 3.2. *Let $f : X \to B$ be a map. Then there is a map $h : C_r \to B$ such that $hi_r = f$ if and only if $fr \sim *$.*

LEMMA 3.3. [15] *Let $g_t : C_r \to B_t (t = 1, 2)$ and $g : C_r \to B_1 \vee B_2$ a map such that $p_t j g i_k \sim g_t i_r (t = 1, 2)$, where $j : B_1 \vee B_2 \to B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \to B_t, t = 1, 2$ are projections. Then there is a map $h : C_r \to B_1 \vee B_2$ such that $gi_r = hi_r$ and $p_t j' h \sim g_t (t = 1, 2)$, where $j' : B_1 \vee B_2 \to B_1 \times B_2$ is the inclusion.*

THEOREM 3.4. *If $X$ is a $DC_k^p$-space with a $DC_k^p$-structure $\theta : X \to A \vee F_k^X$ and $X'$ is a $DC_k^{p'}$-space with a $DC_k^{p'}$-structure $\theta' : X' \to A' \vee F_k^{X'}$ and a map $(s, r) : p' \to p$ is a $DC_k$-map from $\theta'$ to $\theta$, then there is a $DC_k^{\bar{p}}$-structure $\bar{\theta} : C_r \to C_s \vee F_k^{C_r}$ for $C_r$ such that a map $(i_s, i_r) : \theta \to \bar{\theta}$ is a $DC_k$-map from $\theta$ to $\bar{\theta}$.*

*Proof.* Since a map $(s, r) : p' \to p$ is a $DC_k$-map from $\theta'$ to $\theta$, we have that $\theta r \sim (s \vee F_k(r))\theta'$. Thus we know that $(i_s \vee F_k(i_r))\theta r \sim (i_s \vee F_k(i_r))(s \vee F_k(r))\theta' \sim (i_s s \vee F_k(i_r r))\theta' \sim (* \vee *)\theta' \sim * : X' \to C_s \vee F_k^{C_r}$. From Lemma 3.2, there is an extending $\tilde{\theta} : C_r \to C_s \vee F_k^{C_r}$ of $(i_s \vee F_k(i_r))\theta$, that is, $\tilde{\theta} i_r = (i_s \vee F_k(i_r))\theta : X \to C_s \vee F_k^{C_r}$. We see $F_k(i_r) \circ e_k'^X \sim e_k'^{C_s} \circ i_r$ by the naturality of the construction of $F_k^{C_r}$ as is shown in the following homotopy commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{i_r} & C_r \\ e_k'^X\big\downarrow & & e_k'^{C_s}\big\downarrow \\ F_k^X & \xrightarrow{F_k(i_r)} & F_k^{C_r}. \end{array}$$

Then $p_1 j \circ \tilde{\theta} i_r = p_1 j(i_s \vee F_k(i_r)) \circ \theta \sim p_1(i_s \times F_k(i_r)) \circ (p \times e_k'^X)\Delta = i_s \circ p = \bar{p} \circ i_r$ and $p_2 j \circ \tilde{\theta} i_r \sim p_2 \circ (i_s \times F_k(i_r))(p \times e_k'^X)\Delta = F_k(i_r) \circ e_k'^X \sim e_k'^{C_s} \circ i_r$. Thus we have, from Lemma 3.3, that there is a map $\bar{\theta} : C_r \to C_s \vee F_k^{C_r}$ such that $\bar{\theta} i_r = \tilde{\theta} i_r = (i_s \vee F_k(i_r))\theta$ and $p_1 j' \bar{\theta} \sim \bar{p}$, $p_2 j' \bar{\theta} \sim e_k'^{C_s}$. Thus $\bar{\theta} : C_r \to C_s \vee F_k^{C_r}$ is a $DC_k^{\bar{p}}$-structure for $C_r$ such that a map $(i_s, i_r) : \theta \to \bar{\theta}$ is a $DC_k$-map from $\theta$ to $\bar{\theta}$. This proves the theorem. $\square$

Taking $p = 1_X$ and $k = 1$, we can obtain the following corollary.

COROLLARY 3.5. *If $X$ is a co-$T$-space with a co-$T$-structure $\theta : X \to X \vee \Omega\Sigma X$ and $X'$ a co-$T$-space with a co-$T$-structure $\theta' : X' \to X' \vee \Omega\Sigma X'$ and $r : X \to X'$ is a co-$T$-map, then there is a co-$T$-structure on $C_r$ such that $i_r : X \to C_r$ is a co-$T$-map.*

Any co-$H$-space $X$ with a co-$H$-structure $\mu : X \to X \vee X$ is a co-$T$-space $X$ with a co-$T$-structure $\theta = (1 \vee e')\mu : X \to X \vee \Omega\Sigma X$, where $e' = \tau(1_{\Sigma X}) : X \to \Omega\Sigma X$. Also, any co-$H$-map induce a co-$T$-map naturally. Thus we have the above corollary is a generalization of the following proposition.

PROPOSITION 3.6. [6] *If $X$ and $X'$ are co-$H$-spaces, and $r : X' \to X$ is a co-$H$-map, then there is a co-$H$-structure on $C_r$ such that $i_r : X \to C_r$ is a co-$H$-map.*

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